ON A METHOD OF INTEGRATION OF NONSTATIONARY LINEAR BOUNDARY VALUE PROBLEMS ON THE PROPAGATION OF DISTURBANCES IN NON-IDEALLY ELASTIC MEDIA

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In problems of mathematical physics, instead of investigating an ideally elastic model it is sometimes necessary to analyze some others which take into account the imperfections of real materials. When so doing, it is possible to meet the demands of practice without leaving the realm of linear problems. Such problems, for instance, are those of theoretical seismology concerning the propagation of vibrations from a source (explosion), which fit to a high degree of accuracy into the realm of linear processes.

1. Stress-strain relations. Equations of motion. For the sake of brevity, a symbolic notation for the relations, the meaning of which is obvious, will be adopted. Let the relations between the stresses σ and the strains ϵ be given as follows:

$$\sigma = k\varepsilon + K_t\varepsilon \tag{1.1}$$

where k is an elastic constant, and K_t is some linear operator (either a differential operator with respect to time with constant coefficients, or a Volterra type integral operator with respect to time with a differential kernel). Relations (1.1), in the case of an isotropic and homogeneous body which will be investigated here, are characterized by two elastic constants λ, μ (Lame constants) and an operator K_t, which will appear in the relations with different constant coefficients λ', μ'.

Furthermore, it is also assumed that

$$\omega \equiv \frac{\lambda'}{\lambda} = \frac{\mu'}{\mu} \tag{1.2}$$

This is a restriction of the method presented below.

The introduction of additional terms into Hooke's law according to (1.1) takes into account, to some extent, the dissipation of energy in particle oscillations in real matérials.

Therefore, it will be assumed that (1.1) describes a material with dissipation of energy. In this paper such a material will be called a D-medium.

The equations of motion in terms of displacements can be written in a form analogous to Lame's form for the ideally elastic medium, replacing the elastic constants λ, μ by linear (with respect to time) operators:

$$\Lambda = \lambda + \lambda' K_t, \qquad M = \mu + \mu' K_t \tag{1.3}$$

Then the equations of motion of the D-medium become:

$$(\Lambda + 2M)\,\mathrm{grad\,div}\,\mathbf{u} - M\,\mathrm{rot\,rot}\,\mathbf{u} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2} \tag{1.4}$$

where $\mathbf{u} = \mathbf{u}\ (x,\ y,\ z,\ t)$ is the displacement vector written in the Cartesian coordinate system $(x,\ y,\ z)$.

Equations (1.4) can be rewritten in a different form assuming the interchangeability of the operations K_t and differentiation with respect to the coordinates, and applying condition (1.2):

$$(\lambda + 2\mu)\,\mathrm{grad\,div}\,(1 + \omega K_t)\,\mathbf{u} - \mu\,\mathrm{rot\,rot}\,(1 + \omega K_t)\,\mathbf{u} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2} \tag{1.5}$$

Let the displacement vector $\mathbf{u}$ be represented in the usual way by the sum

$$\mathbf{u} = \mathrm{grad}\,\varphi + \mathrm{rot}\,\boldsymbol{\psi} \tag{1.6}$$

where ϕ is a scalar and ψ the vector potential of the displacement field. Then (1.5) can be replaced by an equivalent* system of equations in terms of the unknown functions $\phi\ (x,y,z,t)$ and $\psi\ (x,y,z,t)$:

$$\begin{aligned} (1 + \omega K_t)\,\Delta\varphi &= a^2\frac{\partial^2\varphi}{\partial t^2} \qquad \left(a = \sqrt{\frac{\rho}{\lambda + 2\mu}}\right) \\ (1 + \omega K_t)\,\Delta\boldsymbol{\psi} &= b^2\frac{\partial^2\psi}{\partial t^2} \qquad \left(b = \sqrt{\frac{\rho}{\mu}}\right) \end{aligned} \tag{1.7}$$

* The representation of $\mathbf{u}$ in the form (1.6) is correct within the order of accuracy of the Laplace vector. It is in this sense that the equivalence of (1.5) and (1.7) is understood.

Here a and b are the reciprocals of the velocities of propagation of longitudinal and transverse waves in an elastic medium, respectively. By analogy to the classical theory of elasticity and for the sake of convenience equations (1.7) will be called "wave equations" below.

The symbolic relation (1.1), or, what is the same, the equations of motion (1.5) characterize almost every model of the nonideal theory of elasticity that was proposed until the present time. Actually the latter equations can be divided into two groups:

(a) The first group is characterized by a differential operator K_t, which is linear and has constant coefficients; to this group belong the visco-elastic medium -- the Voigt model [1] -- and the medium with dissipation of energy -- the Newlands model [2].

(b) The second group is characterized by an integral operator K_t of the Volterra type with a differential kernel; to this group belong the model of a medium with an elastic after effect -- the Boltzmann model [3] -- and the model of a medium with relaxation (Maxwell [4]), which actually is contained in the Boltzmann model. These two groups fit completely into the general class of problems examined in this paper.

Since it is not possible to give in a short article a complete survey of work devoted to the investigation of the above mentioned definite models (of group (a) and (b)), only a few papers will be noted here. In the article by Thompson [5] an account of the history of the problem is given, and the general derivation (on the basis of thermodynamic considerations) of the equations of motion of the visco-elastic medium and the medium with an elastic after effect is shown. The studies of Ricker [6] and Voit [7] are devoted to the analysis of nonstationary processes in an infinite visco-elastic medium, described by one wave equation (1.7), with the assumption that the operator is $K_t \equiv \partial/\partial t$.

Dynamic problems for a medium with an elastic after effect were studied by Gogoladze [8] and Deriagin [9]. In this case the operator K_t was determined in the following way:

$$K_t f(t) = \int_{-\infty}^{t} h(t-\tau) f(\tau)\, d\tau \tag{1.8}$$

Here $h(t)$ is an after effect function, and $f(x,y,z,t)$ is a function of the class from which the operator was determined.

The second remark refers to the above mentioned restrictions (condition (1.2) and the presence of only one operator K_t in (1.1)).

The following path of research, evidently, appears to be appropriate in the investigation of dynamic problems for the D-medium. Inasmuch as it is not clear beforehand what operators K_t and parameters λ', μ' should actually be chosen it is above all necessary to attempt to obtain a solution for a wide class of operators, and at least for a few λ', μ' (for example, for those which are subject to (1.2), which is equivalent to the introduction of a parameter ω). Subsequently, the qualitative laws characterizing the features of the wave propagation in D-media for specific K_t (for instance, $K_t = \partial/\partial t$ or (1.8)) should be experimentally verified; and only then it is necessary to go into the accurate investigation of the constructed solutions. From this point of view the restrictions mentioned, do not have such an artificial character as they seemed to have at the beginning.

2. Formulation of the nonstationary problems for non-ideally elastic media. The following problem will be studied. Let U be some region occupied by the D-medium and bounded by the surface S. The oscillations of particles of this body under the action of forces applied on its surface at time $t = 0$ will be studied. Assume that the medium is at rest at $t < 0$, and that the disturbances are produced by the application of the following stresses on the surface S:

$$T_1 = L_1(\mathbf{u})|_S, \qquad T_2 = L_2(\mathbf{u})|_S, \qquad T_3 = L_3(\mathbf{u})|_S \tag{2.1}$$

Here L_1, L_2, L_3 are the usual linear operators in Hooke's law, with the difference that λ, μ are replaced by Λ and M according to formulas (1.3), and the functions T_1, T_2, T_3 are:*

$$T_i = f_i(x, y, z)|_S\, a_i(t) \qquad (i = 1, 2, 3) \tag{2.2}$$

The functions $a_i(t)$ describe the dependence of the reactions T_i on time (in particular, $a_i(t)$ can be the symbolic Dirac function $\delta(t)$, or the Heaviside unit function $\epsilon(t)$; $f_i(x,y,z)/S$ describe the space distribution of the reaction (in the particular problems introduced below the f_i studied will be the function $\delta(r)/r$).

Thus, it is necessary to find $\mathbf{u}(x,y,z,t)$ from the equations of motion (1.4) with zero initial conditions

$$\mathbf{u}|_{t=0} = 0, \qquad \partial\mathbf{u}/\partial t|_{t=0} = 0 \tag{2.3}$$

and boundary conditions (2.1).

Note that because of the linearity of the problem (and for the sake of brevity) the equations will be studied separately for the following boundary conditions:

* The notation $f_i(x,y,z)/S$ denotes that the variables x, y, z are related by the equation $F(x,y,z) = 0$ of the surface S.

$$
\begin{array}{lll}
1) & T_1 \neq 0, & T_2 = T_3 = 0 \\
2) & T_2 \neq 0, & T_1 = T_3 = 0 \\
3) & T_3 \neq 0, & T_1 = T_2 = 0
\end{array}
\tag{2.4}
$$

One of these problems, for example the first one, will be studied more extensively. All discussions can then be simply applied also to problems 2 and 3 of (2.4). Problem (1.4) - (2.3) - (2.1) for $K_t \equiv 0$ and with t replaced by τ will be called the first auxiliary problem.

Thus, the solution of the nonstationary problem of the elastic medium, where the boundary reactions are given as before in the form (2.2) and all $a_i(t)$ are replaced by $\delta(\tau)$, will serve as an auxiliary solution.

The following one-dimensional problem will be studied as the second auxiliary problem:

$$
(1 + \omega K_t)\frac{\partial^2 R(t, \tau)}{\partial \tau^2} = \frac{\partial^2 R(t, \tau)}{\partial t^2}
$$

$$
R\Big|_{t=0} = \frac{\partial R}{\partial t}\Big|_{t=0} = 0, \qquad (1 + \omega K_t) R(t, \tau)\,|_{\tau=0} = a(t) \tag{2.5}
$$

Theorem. The solution of problem (1.4) - (2.3) - (2.1) is given by the formula

$$
\mathbf{u}(x, y, z, t) = \int_0^\infty \mathbf{u}_0(x, y, z, \tau) R(t, \tau)\, d\tau \tag{2.6}
$$

where $\mathbf{u}_0(x,y,z,\tau)$ is the solution of the first auxiliary problem, and $R(t,\tau)$ is the solution of problem (2.5).

Proof. Substituting $\mathbf{u}(x,y,z,t)$ in form (2.6) into the equation of motion (1.5) and allowing for the possibility of repeated integration with respect to the coordinates under the integral sign yields

$$
\int_0^\infty (1 + \omega K_t) R(t, \tau)\, [(\lambda + 2\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}_0(x, y, z, \tau) -
$$

$$
- \mu\,\mathrm{rot}\,\mathrm{rot}\,\mathbf{u}_0(x, y, z, \tau)]\, d\tau = \rho \int_0^\infty \frac{\partial^2 R(t, \tau)}{\partial t^2} \mathbf{u}_0(x, y, z, \tau)\, d\tau
$$

Inasmuch as $\mathbf{u}_0$ appears as the solution of the auxiliary problem (i.e. satisfies the Lame equations) then

$$
\rho \int_0^\infty (1 + \omega K_t) R(t, \tau) \frac{\partial^2 \mathbf{u}_0(x, y, z, \tau)}{\partial \tau^2}\, d\tau = \rho \int_0^\infty \frac{\partial^2 R(t, \tau)}{\partial t^2} \mathbf{u}_0(x, y, z, \tau)\, d\tau
$$

Use the equality

$$\int_0^\infty R(t,\ \tau)\frac{\partial^2 \mathbf{u}_0(x,\ y,\ z,\ \tau)}{\partial\tau^2}\,d\tau = \int_0^\infty \frac{\partial^2 R\ (t,\ \tau)}{\partial\tau^2}\,\mathbf{u}_0(x,\ y,\ z,\ \tau)\,d\tau \tag{2.7}$$

which is obtained by integrating twice by parts.

In addition, it was necessary to assume that the following conditions (of the type of the radiation condition) be satisfied:

$$R^2\frac{\partial}{\partial\tau}\left[\frac{\mathbf{u}_0}{R}\right]\to 0 \qquad \text{as}\quad \tau\to\infty \tag{2.8}$$

The last conditions are usually automatically satisfied in the problems at hand. Then $\mathbf{u}(x,y,z,t)$ satisfies (1.5) by virtue of the equation for $R(t,\tau)$.

The initial conditions for $\mathbf{u}(x,y,z,t)$ represented in the form (2.6) are satisfied because of the zero initial conditions for $R(t,\tau)$. In verifying the boundary conditions one starts from the fact that the basic problem is divided into three problems according to (2.4), and only the solution for the first of them is to be verified. Substitute $\mathbf{u}(x,y,z,t)$ in the form (2.6) into (2.1). Assuming that it is proper to carry out the operation L_1 under the integral on τ, the following expression for the first of the boundary conditions is obtained:

$$f_1(x,\ y,\ z)\,|_S\, a(t) = \int_0^\infty L_1[\mathbf{u}_0(x,\ y,\ z,\ \tau)]\,\Big|_S (1+\omega K_1)\,R(t,\ \tau)\,d\tau \tag{2.9}$$

If one considers that

$$L_1[\mathbf{u}_0(x,\ y,\ z,\ \tau)]\,|_S = f_1(x,\ y,\ z)\,|_S\,\delta(\tau)$$

(see the first auxiliary problem), then (2.9) is obviously satisfied by virtue of the boundary condition for $R(t,\tau)$ at $\tau = 0$ and for an arbitrary $f_1(x,y,z)/S$. In an analogous fashion also the boundary conditions of problems 2 and 3 of (2.4) can be verified. Thus the basic theorem is proved.

Remark 1. The solution $\mathbf{u}_0(x,y,z,\tau)$ for many interesting problems of the dynamic theory of elasticity can be formed either by the method of functional-invariant solutions of Smirnov and Sobolev [10], or by the method of the incomplete separation of variables, proposed by Smirnov [11] and developed by Petrashen' [12, 13].

In such a way, for instance, solutions were formed for the elastic half-space under nonstationary boundary point loads [13, 14], for layerwise-isotropic media with parallel-plane separation boundaries [15], and for media with spherical or cylindrical separation boundaries [16].

Remark 2. It is necessary to show a method of forming and analyzing the function $R(t,\tau)$, which will be done in the following paragraph, in order to consider the solution of these problems $\mathbf{u}(x,y,z,t)$ as being determined.

In the same fashion the basic formula (2.6) allows the construction of solutions of the boundary value problems, mentioned in the first remark, also for a D-medium (with some restrictions on the operator K_t of the same type as before). In this sense the basic formula contains the general qualitative result on the construction of solutions of dynamic problems for D-media with the aid of already known solutions $\mathbf{u}_0(x,y,z,\tau)$ and some special functions $R(t,\tau)$.

3. The Functions $R(t,\tau)$. Let us study problem (2.5). By virtue of the zero initial conditions of the problem the solution can be sought in the form

$$R(t, \tau) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} R_0(s, \tau) e^{st} ds \qquad (\operatorname{Re} s = \sigma > 0)$$

and the application of the operator K_t to $R(t,\tau)$ leads to the formula

$$K_t R(t, \tau) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} R_0(s, \tau) K(s) e^{st} ds \quad \left(K(s) = \int_0^\infty e^{-st} K_t \delta(t)\, dt\right) \quad (3.1)$$

Without dwelling on elementary calculations connected with the finding of $R_0(s,\tau)$ form problem (2.5), the final result is immediately written as follows:

$$R(t, \tau) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{A(s)}{1+\omega K(s)} \exp\left(st - \frac{s\tau}{\sqrt{1+\omega K(s)}}\, ds\right) \qquad (3.2)$$

$$\left(A(s) = \int_0^\infty a(t) e^{-st} dt\right)$$

Here $K(s)$ comes from (3.1), and the contour of integration is the straight line Re $s = > 0$, parallel to the imaginary axis in the plane of the complex variable s; the branch cut of the radical $\sqrt{1 + \omega K(s)}$ is fixed by the condition

$$\arg\sqrt{1+\omega K(s)} = 0 \qquad \text{for } s > 0$$

The solution can be verified by direct substitution of (3.2) into (2.5), where it is necessary to take into account relation (3.1). When doing so, it is not necessary to consider the question of the appropriateness of differentiating under a contour integral and going to the limit in the verification of the initial and boundary conditions inasmuch as all those operations are proper from the point of view of the theory of generalized functions, with which these problems deal.

Application of Borel's theorem to (3.2) yields

$$R(t,\ \tau)=\int_0^t a(t_1)\,R_\delta(t-t_1,\ \tau)\,d\tau$$

Then also (2.6) can be rewritten as

$$\mathbf{u}(x,\ y,\ z,\ t)=\int_0^t a(t_1)\Big[\int_0^\infty \mathbf{u}_0(x,\ y,\ z,\ \tau)\,R_\delta(t-t_1,\ \tau)\,d\tau\Big]dt_1 \tag{3.3}$$

From the latter it obviously follows that the study of the function $R_\delta(t,\tau)$ is of basic interest:

$$R_\delta(t,\ \tau)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{1}{1+\omega K(s)}\exp\left(st-\frac{s\tau}{\sqrt{1+\omega K(s)}}\right)ds \tag{3.4}$$

Note, first of all, that $R_\delta(t,\tau)\to\delta(t-\tau)$ as $\omega\to 0$. Then, in the limit $\omega=0$, formula (3.3) gives the solution of the elastic problem $\mathbf{u}_0(x,y,z,t)$ with a load that varies with time as $a(t)$.

In particular, if a sufficiently smooth function is chosen for $a(t)$, then the classical sense of the limit solutions as well as the limiting process $\omega\to 0$ itself can be justified.

Note the essential property of the integrand in $R(t,\tau)$. The expression $1+\omega K(s)$ (and with it also the entire integral in (3.2)) does not have any singularities in the right hand half-plane of the complex variable s, including the imaginary axis. For the integral operators K_t this follows directly from certain inequalities obtained from energy considerations; for the differential operators K_t this is in most cases an obvious fact.

Using this feature of the integrand in (3.4) the integral $R_\delta(t,\tau)$ can be given a real representation in the form of a Fourier integral. For this purpose it is necessary to deform the contour of integration $\operatorname{Re} s=\sigma>0$ into the imaginary axis and change the variable of integration in (3.4): $s=i\lambda$. Then

$$R_\delta(t,\ \tau)=\frac{1}{\pi}\int_0^\infty e^{\vartheta}\cos\left[\lambda\left(t-\frac{\theta}{\lambda}\right)-\varphi\right]\frac{d\lambda}{\rho} \tag{3.5}$$

where

$$1+\omega K(i\lambda)=u+iv=\rho e^{i\varphi},\qquad \vartheta=\exp\left(-\frac{\lambda\tau}{\sqrt{\rho}}\sin\frac{\varphi}{2}\right),\qquad \theta=\frac{\lambda\tau}{\sqrt{\rho}}\cos\frac{\varphi}{2}$$

The presence of the factor e^{ϑ} in the integrand and the item θ in the cosine expression indicates damping and dispersion with the propagation of disturbances in D-media.

4. The solution of the nonstationary problem for the visco-elastic half-space. In this case the operator is $K_t = \partial/\partial t$, and as a consequence equations (1.5) and the boundary conditions are simplified.

The solution $\mathbf{u}(r,\theta,z,t)$ of equation (1.5) with zero initial conditions (2.3) and concentrated loading on the boundary is to be found* in the half-space $z > 0$, filled with a visco-elastic material. The statement on the boundary conditions implies that at $z = 0$ at the point $r = 0$ the following functions are given:

$$\text{a)}\quad T_{zz} = \frac{\delta(r)}{r} a(t), \quad T_{rz} = 0 \, (r = \sqrt{x^2 + y^2}) \quad \left(\begin{matrix}\text{axisymmetric}\\ \text{loading}\end{matrix}\right) \tag{4.1}$$

$$\text{b)}\quad T_{xz} = \frac{\delta(r)}{r} a(t), \quad T_{zz} = T_{yz} = 0 \quad \left(\begin{matrix}\text{non-symmetric}\\ \text{tangential loading}\end{matrix}\right) \tag{4.2}$$

In the case of the visco-elastic material the second auxiliary function has the form:**

$$R_\delta(t, \tau) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{1}{1 + \omega s} \exp\left(st - \frac{s\tau}{\sqrt{1 + \omega s}}\right) ds, \qquad \operatorname{Re} s = \sigma > 0 \tag{4.3}$$

In order to obtain the solution of the given problem according to the basic formula (2.6), it is necessary to know $\mathbf{u}_0(r,\theta,z,\tau)$. The latter can be obtained, for instance, by a formal differentiation with respect to t of the solutions of the dynamics problems for the elastic half-space [13, 14].

In the case of an axisymmetric leading (4.1) $\mathbf{u}_0(r,\theta,z,\tau)$ has the form

$$\mathbf{u}_0 = u_{0r}\mathbf{r}_0 + u_{0z}\mathbf{k}$$

$$u_{0r} = \frac{1}{2\pi\mu b} \int_0^\infty \left\{ \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \left[\frac{2 + \zeta^2}{R(\zeta)} g(\gamma, \zeta) - \frac{2\sqrt{1 + \zeta^2}\sqrt{1 + \gamma^2\zeta^2}}{R(\zeta)} g(1, \zeta) \right] d\zeta \right\} kJ_1(kr)\, dk$$

$$u_{0z} = \frac{1}{2\pi\mu b} \int_0^\infty \left\{ \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \left[\frac{(2 + \zeta^2)\sqrt{1 + \gamma^2\zeta^2}}{R(\zeta)} g(\gamma, \zeta) - \frac{2\sqrt{1 + \gamma^2\zeta^2}}{R(\zeta)} g(1, \zeta) \right] d\zeta \right\} kJ_0(kr)\, dk \tag{4.4}$$

$$g(\gamma, \zeta) = \exp\left[k\left(\zeta \frac{\tau}{b} - z\sqrt{1 + \gamma^2\zeta^2}\right)\right]$$

* For considerations of convenience the cylindrical coordinate system (r,θ,z) related to the Cartesian coordinate system (x,y,z) by the usual transformation formulas is chosen.

** An analogous integral was studied by numerical methods in the paper by Zverev [19].

where

$$\gamma = \frac{a}{b}, \qquad M_a = \sqrt{1+\gamma^2\zeta^2}, \qquad M_b = \sqrt{1+\zeta^2}$$
$$R(\zeta) = (2+\zeta^2)^2 - 4M_aM_b, \qquad \operatorname{Re}\zeta = \sigma > 0 \tag{4.5}$$
$$\arg M_a = \arg M_b = 0 \qquad \text{for } s>0$$

The solution of the problem with boundary condition (4.2) has an analogous representation [14].

Consider briefly some physical consequences of the obtained solutions. For this purpose $\mathbf{u}_0(r,\theta,z,\tau)$ and $R(t,\tau)$ will be analyzed separately, and the product of these functions will be integrated. As a result of this, according to the basic formula (2.6), a representation of the behavior of the solution of the problem $\mathbf{u}(r,\theta,z,t)$ will be obtained. For the sake of brevity only an isolated typical expression, which is a part in the description of the displacement field, will be investigated here.

At first $\mathbf{u}_0(r,\theta,z,\tau)$ is to be analyzed. If one uses the asymptotic methods of analysis (for large τ) of the elastic problem, as shown in [12, 13], then one can give the following representation of the main parts of the displacement field, characterizing volumetric waves.

The term

$$-\frac{1}{2\pi\mu b}\int_0^\infty \left\{\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{2\sqrt{1+\gamma^2\zeta^2}}{R(\zeta)}\, g(1,\ \zeta)\, d\zeta\right\} kJ_0(kr)\, dk \tag{4.6}$$

taken from (4.4) (from the u_{0z} component) characterizes the singularity of the solution in the neighborhood of the surface $\tau^2 = b^2(r^2+z^2)$ and has there the following principal part:

$$-\frac{z}{2\pi\mu\sqrt{r}}\left[\frac{b}{(\tau^2-b^2z^2)^{3/2}}\right]^{1/2} \frac{\sqrt{|1+\gamma^2\zeta_b{}^2|}}{R(\zeta_b)}\,\delta\left(\frac{1}{b}\sqrt{\tau^2-b^2z^2}-r\right)$$
$$\left(\zeta_b = \frac{i\tau}{\sqrt{\tau^2-b^2z^2}}\right) \tag{4.7}$$

All remaining terms from (4.4) can be analyzed in a similar fashion. In addition one can also investigate in the usual way the surface waves and conical waves, only that the latter will subsequently require a numerical integration. Of main interest are, however, the volumetric (longitudinal and transverse) waves described with the aid of (4.7). For a complete description of the volumetric waves it is necessary to carry out the integration with respect to τ of the results (4.7) with a function $R_\delta(t,\tau)$ corresponding to the basic formula. If it is assumed that a simple integration formula for $\delta(\xi)$ can be derived

$$\int_{\alpha}^{\beta} f(\tau,\ r,\ \theta,\ z)\,\delta[\xi(\tau)]\,d\tau = \frac{f(\tau_0,\ r,\ \theta,\ z)}{\xi'(\tau_0)} \qquad \begin{pmatrix}\xi(\tau_0)=0\\ \alpha<\tau_0<\beta\end{pmatrix} \tag{4.8}$$

then the final result for the reduced component displacement field can be represented in the following way: the component (axisymmetric reaction)

$$\begin{gathered} u_z = \frac{z}{2\pi\mu r^2}\frac{1}{\sqrt{r^2+z^2}}\left[\frac{\sqrt{|1+\gamma^2\zeta_b{}^2|}}{R(\zeta_b)}R_\delta(t,\ \tau)\right]_{\tau=\tau_0} \\ \zeta_b = \frac{i\sqrt{r^2+z^2}}{r}, \qquad \tau_0 = b\sqrt{r^2+z^2} \end{gathered} \tag{4.9}$$

In this fashion everything is reduced to the analysis of the function $R(t,r)$ [or $R_\delta(t,z,)$] according to (3.3), where $r = a\sqrt{r^2+z^2}$ in the case of longitudinal waves, and $r = b\sqrt{r^2+z^2}$ in the case of transverse waves.

As was shown at the end of Section 3, the integral $R_\delta(t,r)$ can be represented as a Fourier integral

$$R_\delta(t,\ \tau) = \frac{1}{\pi}\int_0^\infty \exp\left(-\frac{\lambda\tau}{\sqrt{\rho}}\sin\frac{\varphi}{2}\right)\cos\left[\lambda\left(t-\frac{\tau}{\sqrt{\rho}}\cos\frac{\varphi}{2}\right)-\varphi\right]\frac{d\lambda}{\rho} \tag{4.10}$$

$$\rho = \sqrt{1+\omega^2\lambda^2}, \qquad \varphi = \operatorname{arc\,tg}\omega\lambda$$

The damping and dispersion of waves in the stationary region are characterized, respectively, by the functions

$$\exp\left(-\frac{\lambda\tau}{\sqrt{\rho}}\sin\frac{\varphi}{2}\right), \qquad \frac{1}{\sqrt{\rho}}\cos\frac{\varphi}{2} \tag{4.11}$$

If the values of r are substituted (for the longitudinal and transverse waves separately) into (4.11), then the damping and the dispersion can be compared for longitudinal and transverse waves.

The approximate expressions of the functions (4.11) are to be studied. For $\omega\lambda < 1$ we obtain $\exp(-\tfrac{1}{2}\lambda^2\omega r)$ in first approximation the damping and no dispersion (the latter indicates that for frequencies $\lambda < \omega^{-1}$ the dispersion appears as an effect of a higher order in comparison to the damping). For $\omega\lambda > 1$ the damping and the dispersion are of the same order. An experimental determination of the magnitude of ω, as an indication of the qualitative bound on the frequency scale, can serve as a check on the usefulness of the model analyzed for the description of dynamic processes in real materials.

The function $R_\delta(t,r)$ can be also expressed in an approximate fashion

$$R_\delta(t,\ \tau) \approx \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\omega\tau}}\exp\left(-\frac{(t-\tau)^2}{2\omega\tau}\right) \tag{4.12}$$

The last representation is convenient for the study of nonstationary phenomena. Actually, (4.12) indicates* the following effects: (1) at a fixed point in space, the maximum disturbance is reached at $t = r$, i.e. in a visco-elastic material the disturbances propagate with the velocities $1/a$ and $1/b$, respectively; (2) for impulsive loads the wave is dome-shaped, and the more sharply defined (for fixed r) the smaller the parameter ω; (3) the shape of the wave acquires a more and more "washed out" character with distance (the "diffusion" is proportional to the square root of the distance travelled by the maximum disturbance); (4) there is additional damping of waves because of dissipation of energy, which is indicated by the presence of the factor $(\omega r)^{-\frac{1}{2}}$ in (4.12).

The experimental study of the gradual change of shape of a wave of one type can indicate a method for determining the magnitude of ω for real materials.

Finally, note that if the condition (1.2) is removed and an additional notation is introduced

$$\omega_1 = \frac{\lambda' + 2\mu'}{\lambda + 2\mu}$$

then the formulas for the principal parts of (4.9) should be changed only for the longitudinal waves by the replacement of $R_\delta(t, r)$ by

$$R_{\delta p}(t, \tau) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{1}{1 + \omega s} \exp\left(st - \frac{s\tau}{\sqrt{1 + \omega_1 s}}\right) ds \tag{4.13}$$

Note also that from physical considerations $\omega_1 < \omega$ (this corresponds also to experimental data).

5. Solution of the nonstationary problem for a half-space occupied by a medium with an elastic after-effect. In this case the operator K_t is determined, as shown in Section 1 by the relation (1.9). The function $h(t - r)$ is unknown. It is necessary to determine it from experimental data on physical characteristics of propagation of waves in real materials. Here one can proceed in two ways: One can assume beforehand a definite form for $h(t - r)$, find the solution of the problem, and then by comparing with experimental data choose a suitable $h(t - r)$. A second way will be chosen here, however; namely, leaving $h(t - r)$ unknown, the following simple conditions will be imposed on it:

$$\begin{aligned} &(a) \quad h(t, r) \equiv h(t - r) \quad \text{(V. Volterra)} \\ &(b) \quad h(t) \quad \text{is a positive and decreasing function as } t \to \infty \\ &(c) \quad \omega \int_0^\infty h(t)\, dt < 1 \end{aligned} \tag{5.1}$$

* It should be noted that in (4.12) r should be replaced by $a\sqrt{r^2 + z^2}$ for longitudinal waves and by $b\sqrt{r^2 + z^2}$ for transverse waves.

(the last inequality was obtained by Gogoladze [8] from the condition that the potential energy of deformation in the material analyzed here is positive).

In this paragraph it is pointed out that it is in principle possible to determine $h(t)$ from experimental data on the propagation of waves.

The only difference between the solution of the problem for the half-space occupied by a material with an elastic after effect and the problem of Section 4 consists in the fact that in the basic formula (2.6), which gives the solution to the problem, the $\mathbf{u}_0(r,\theta,z,\tau)$ is to be taken from (4.4), and for $R_1(t,\tau)$ one should have

$$R_1(t,\ \tau) = \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} \frac{A(s)}{1-\omega H(s)} \exp\left(st - \frac{s\tau}{\sqrt{1-\omega H(s)}}\right) ds \tag{5.2}$$

Here $A(s)$ comes from (3.2)

$$\operatorname{Re} s = \sigma > 0, \qquad \arg\sqrt{1-\omega H(s)} = 0 \qquad \text{for } s>0$$

$$H(s) = \int\limits_0^\infty h(t)\, e^{-st}\, dt$$

Analyze closer the following function:

$$R_{1\delta}(t,\ \tau) = \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} \frac{1}{1-\omega H(s)} \exp\left(st - \frac{s\tau}{\sqrt{1-\omega H(s)}}\right) ds \tag{5.3}$$

The integrand in (5.3) is regular in the right-hand half-plane of the compex variable s, including the imaginary axis. The latter is true by virtue of the inequality

$$\omega\,|H(s)| < 1 \tag{5.4}$$

which follows from conditions (b) and (c) on $h(t)$ in (5.1). Then the previous method of analysis can be applied to (5.3) by reducing the problem to the Fourier integral. As a result of simple transformations one obtains

$$R_{1\delta}(t,\ \tau) = \frac{1}{\pi}\int\limits_0^\infty \exp\left(-\frac{\lambda\tau}{\sqrt{\rho_1}}\sin\frac{\varphi_1}{2}\right)\cos\left[\lambda\left(t - \frac{\tau}{\sqrt{\rho_1}}\cos\frac{\varphi_1}{2}\right) - \varphi_1\right]\frac{d\lambda}{\rho_1} \tag{5.5}$$

$$\rho_1 e^{i\varphi_1} = 1 - \omega H(i\lambda) = 1 - \omega u_1 + \omega v_1 \qquad (H(i\lambda) = u_1 - iv_1)$$

By virtue of the inequality (5.4) the integral in (5.5) can be given the following approximate representation:

$$R_{1\delta}(t,\ \tau) = \frac{1}{\pi}\int\limits_0^\infty \exp\left(-\frac{\lambda\omega\tau}{2}v_1\right)\cos\lambda\left(t - \tau - \frac{\omega\tau}{2}u_1\right) d\lambda$$

The quantities $\exp(-\frac{1}{2}\lambda\omega r v_1)$ and $\frac{1}{2}\omega r u_1$, can be determined experimentally, and hence $u_1(\lambda)$ and $v_1(\lambda)$ will be known, after which one can use the following relation for finding $h(t)$

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(i\lambda) e^{i\lambda t}\, d\lambda$$

The latter was obtained from the Mellin inversion formula for $H(s)$ by going over to the Fourier integral (which is correct because of the inequality (5.4)).

Conclusion.

First, the method of the incomplete separation of variables was used in the construction of the solutions of specific problems in Sections 4 and 5 where the system of "wave" equations (1.7) served as the basis. The subsequent application of the transformation of Efros [18] facilitated the derivation of the basic formula (2.6) for the solutions obtained. Further generalizations, for the class of problems for D-media shown above, as well as for other nonstationary boundary value problems (for layered isotropic D-media with plane parallel separation boundaries; for regions occupied by the D-medium and having spherical or cylindrical separation boundaries) were obtained with the aid of a basic formula. Some restrictions on the operator K_t, characterizing the D-medium, of the type shown in the main text of the article as well as taking results from [15, 16] as auxiliary material for the construction of $\mathbf{u}_0(x,y,z,r)$ in corresponding problems, was all that was necessary for this generalization.

Note also that if the above mentioned restrictions are removed, one can construct a solution by the method of incomplete separation of variables for the enumerated problems in the case of D-media, with the difference that then a direct analysis appears to be difficult. In some simple problems (for instance, for a half-space occupied by a D- medium) the analysis can be performed with the aid of the generalization of the Efros transformation and a symptotic method of analyzing disturbance fields. In particular, formula (4.13) was obtained in just this way.

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